C. Woodward – 28/5/08 – Floer field theory (w/ Wehrheim)

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Mirror symmetry perspective:

$$\text{symplectic geometry} \xleftrightarrow{\text{M.S.}} \text{complex geometry}$$

$$DFuk(M) \xleftrightarrow{\text{HMS}} DCoh(W)$$

$$\underline{\text{quilt functors}} \xleftrightarrow{\text{2-HMS}} \underline{\text{Mukai functors}}$$

$L_{01} \subset M_0^- \times M_1$ $\qquad\qquad$ $Z_{01} \in Ob(DCoh(W_0 \times W_1))$

$\rightsquigarrow \phi(L_{01}) : DFuk^{\#}(M_0) \to DFuk^{\#}(M_1)$ $\qquad$ $\rightsquigarrow \phi(Z_{01}) : DCoh(W_0) \to DCoh(W_1)$

extended Fukaya cat's

objects = generalized correspondences

$$pt \xrightarrow{L_{-k}} M_{-k} \xrightarrow{L_{-k+1}} \cdots \to M$$

## Last time:

**Main thm:**

$L_{01} \subset M_0^- \times M_1$ $\quad$ monotone, min. Maslov $\geq 3$,

$L_{12} \subset M_1^- \times M_2$ $\quad$ $L_{01} \circ L_{12}$ smooth and embedded

$\Rightarrow$ then $\phi(L_{01} \circ L_{12}) \simeq \phi(L_{01}) \circ \phi(L_{12})$

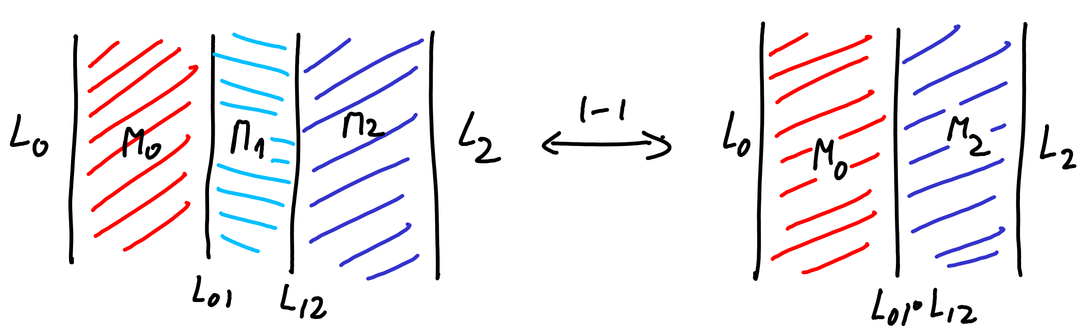
**Concretely:**

$L_0 \subset M_0$, $L_2 \subset M_2$ branes

$\Rightarrow HF(L_0 \times L_{12}, L_{01} \times L_2) \simeq HF(L_0 \times L_2, L_{01} \circ L_{12})$

in $M_0 \times M_1 \times M_2$ $\qquad\qquad$ in $M_0 \times M_2$

**Idea of pf.** compare quilted trajectories



Today: application of Floer field theory for 3-dim$^{\underline{l}}$ cobordisms
(possibly with graphs; rank & degree coprime)

History: Floer ⇝ instanton homology for homology 3-spheres
Donaldson ⇝ relative 4-mfld invariants
Fukaya ⇝ proposal for Floer invariants for cobordisms

Motivation: (1) this was the original motivation for Fukaya categories
(2) do it with graphs (or links... generalize to graphs so can get exact sequences).

Conjectural picture:

$$\begin{tikzcd}
\text{Instanton knot homology (Collin-Steer, Kronheimer-Mrowka)} \arrow[d, dashed, leftrightarrow, "\text{Atiyah-Floer (rank } r)"] & \text{Khovanov-Rozansky homologies} \arrow[d, dashed, leftrightarrow] \\
\text{Lagrangian Floer homology} \arrow[r, dashed, leftrightarrow, "\text{spectral sequence}"'] & \text{Seidel-Smith-Manolescu (sympl. Khovanov-Rozansky)}
\end{tikzcd}$$

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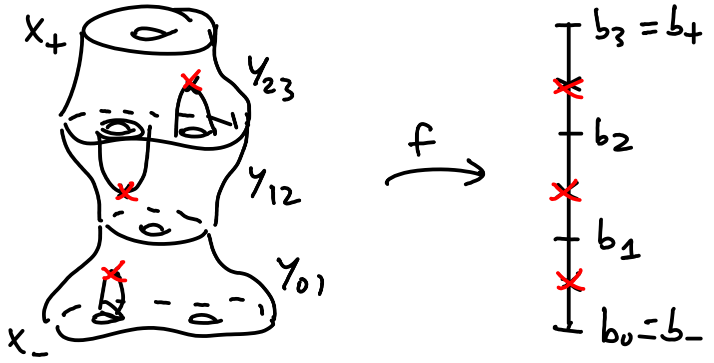
TFT via Cerf theory:

$X_\pm$ d-manifolds (compact, oriented)

$Y$ cobordism from $X_-$ to $X_+$

A Morse datum for $Y$ is a pair $(f, \underline{b})$ where

$$\begin{cases} f: Y \longrightarrow [b_-, b_+] \text{ Morse, with distinct crit. vals. } \in (b_-, b_+) \\ X_\pm = f^{-1}(b_\pm) \\ \underline{b} = (b_0, \dots, b_m) \in \mathbb{R}^{m+1} \text{ separates the critical values } (b_0 = b_-,\ b_m = b_+) \end{cases}$$



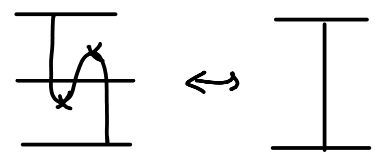

$Y = Y_{01} \cup \dots \cup Y_{(m-1)m}$ Cerf decomposition for $(f, \underline{b})$.

<u>Thm</u> (Thom, Mather, Cerf)

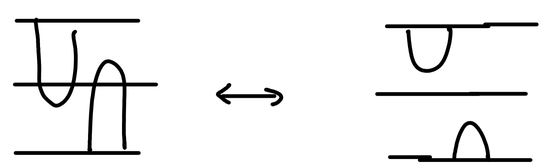
Any two Cerf decompositions of $Y$ are related by

(i) diffeomorphism

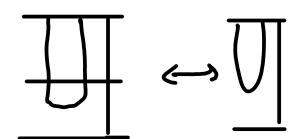
(ii) critical point cancellation



(iii) order reversal



(iv) appendectomies = remove trivial pieces



<u>NB</u>: don't see handle slides here because we're doing critical pts one at a time ( <u>simple</u> cobordisms = $\leq 1$ crit pt)

[only need $f$ Morse, not Morse-Smale ! in handlebody pictures need $f$ Morse-Smale to define all attaching cycles, handle slide = transition through Morse fn that isn't Smale]

<u>Corollary:</u>

$\mathcal{C}$-valued TFT's
i.e. functors
$Cob_{d+1} \to \mathcal{C}$
$\uparrow$ $\otimes$-category

$\xleftrightarrow{1-1}$

data
$\begin{cases} d\text{-manifolds} \to ob(\mathcal{C}) \\ \text{simple cobordisms} \to \text{morphisms} \\ \quad \text{satisfying Cerf relations} \end{cases}$

# Moduli of bundles of coprime rank & degree $(r,d)$:

- A **decorated surface** consists of

$$\begin{cases} \text{a surface } X, \quad \underline{\text{connected}} \\ \text{a principal } U(r) \text{ bundle } P \to X \quad \text{with degree } d = \langle c_1(P), [X] \rangle \\ \text{a connection } \delta \text{ on } \det(P) \end{cases}$$

given a decorated surface $X$, define

$$M(X) = \left\{ \begin{array}{l} A = \text{connections on } P \text{ w/ central curvature} = R^{\delta} \\ \text{and s.t. } \det(A) = \delta \end{array} \right\} \Big/ \text{gauge}$$

$$\cong \left\{ (A_1, B_1, \dots, A_g, B_g) \in SU(r) \text{ s.t. } \prod_{j=1}^{g} [A_j, B_j] = \operatorname{diag}\left(\exp\left(\frac{2\pi i d}{r}\right)\right) \right\} \Big/ SU(r)$$

(here $g = \text{genus}(X)$)

Seshadri et al $\Rightarrow$ $M(X)$ is a compact, smooth, monotone symplectic manifold.

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Let $Y^3$ = cobordism from $X_-$ to $X_+$ (connected)

Assume $Y$ is decorated, ie. extensions $(P, \delta)$ of $(P_\pm, \delta_\pm)$ to $Y$ are given

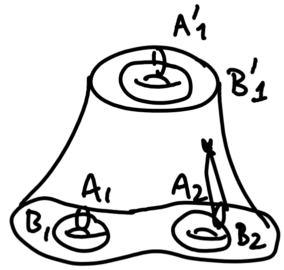
Then $M(Y) \longrightarrow M(X_-)^- \times M(X_+)$ has Lagrangian image $=: L(Y)$

$A \longmapsto (A_{|X_-}, A_{|X_+})$

⚠ [Lagr. correspondence ?]

<u>Thm:</u> $Y$ simple (at most 1 crit. pt, of index 1 or 2) $\Rightarrow$ $L(Y)$ is a smooth monotone brane.

Ex:




$\rightsquigarrow \ L(Y) = \left\{ \begin{array}{l} A_1 = A_1', \quad B_1 = B_1', \\ A_2 = Id, \quad B_2 = \text{anything} \end{array} \right\}$

★ Let $Y$ = arbitrary decorated cobordism from $X_-$ to $X_+$

$(f, \underline{b})$ Morse datum with connected level sets

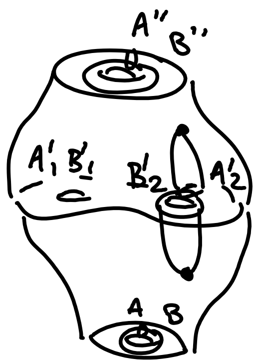
$\underline{Y} = (Y_{01} \cup \dots \cup Y_{(m-1)m})$ Cerf decomposition

$L(\underline{Y}) := L(Y_{01}) \# L(Y_{12}) \# \dots \# L(Y_{m-1,m})$ (generalized correspondence)

→ Thm: $\left[ \begin{array}{l} X \to M(X) \\ Y \to L(\underline{Y}) \end{array} \right]$ is a TFT with values in $\left[ \begin{array}{c} \text{sympl. manifolds} \\ \text{gen}^d\text{. Lag. correspondences} \end{array} \right]$

(up to geom. ~ algebraic compositions)

PF: Need to check Cerf moves: e.g. crit pt cancellation:




$L(Y_{12}) = \{A_1' = A'', \ B_1' = B'', \ A_2' = Id\}$

$L(Y_{01}) = \{A_1' = A, \ B_1' = B, \ B_2' = Id\}$

$L(Y_{01}) \circ L(Y_{12}) = \left\{ \begin{array}{l} A = A'' \\ B = B'' \end{array} \right\} = \Delta \subset L(X \times [0,1])$

similarly for other moves ....

▲

Problem: Compute mirror of this TFT?

ie: $X^2$ surface (decorated) $\rightsquigarrow$ some LG-model (mirror to $M(X)$)

cobordisms $\longmapsto$ Mukai correspondences on $D^b Sing$'s?